The First Derivative of Ramanujans Cubic Continued Fraction

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Abstract

We give the complete evaluation of the first derivative of the Ramanujans cubic continued fraction using Elliptic functions. The Elliptic functions are easy to handle and give the results in terms of Gamma functions and radicals from tables.

keywords Ramanujan's Cubic Fraction; Jacobian Elliptic Functions; Continued Fractions; Derivative

1 Introduction

The Ramanujan's Cubic Continued Fraction is (see [3], [7], [8], [9], [11]).

$$V(q) := \frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\ldots \tag{1}$$

Our main result is the evaluation of the first derivative of Ramanujan's cubic fraction. For this, we follow a different way from previous works and use the theory of Elliptic functions. Our method consists to find the complete polynomial equation of the cubic fraction which is a solvable, in radicals, quartic equation, in terms only of the inverse elliptic nome k_r, Using the derivative of k_r which we evaluate in Section 2 of this article, we find the desired formula of the first derivative. For beginning we give some definitions first.
Let

$$(a;q)_k = \prod_{n=0}^{k-1}(1-aq^n) \tag{2}$$

Then we define

$$f(-q) = (q;q)_\infty \tag{3}$$

and

$$\Phi(-q) = (-q;q)_\infty \tag{4}$$

Also let

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1-x^2\sin^2(t)}}dt \tag{5}$$

be the elliptic integral of the first kind.
We denote

$$\theta_4(u,q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2nui} \tag{6}$$

the Elliptic Theta function of the 4th-kind. Also hold the following relations (see [16]):

$$\prod_{n=1}^{\infty}(1-q^{2n})^6 = \frac{2kk'K(k)^3}{\pi^3 q^{1/2}} \tag{7}$$

$$q^{1/3}\prod_{n=1}^{\infty}(1+q^n)^8 = 2^{-4/3}\left(\frac{k}{1-k^2}\right)^{2/3} \tag{8}$$

and

$$f(-q)^8 = \prod_{n=1}^{\infty}(1-q^n)^8 = \frac{2^{8/3}}{\pi^4}q^{-1/3}k^{2/3}(k')^{8/3}K(k)^4 \tag{9}$$

The variable k is defined from the equation

$$\frac{K(k')}{K(k)} = \sqrt{r} \tag{10}$$

where r is positive , $q = e^{-\pi\sqrt{r}}$ and $k' = \sqrt{1-k^2}$. Note also that whenever r is positive rational, the $k = k_r$ are algebraic numbers.

2 The Derivative $\{r, k\}$

Lemma 1.
If $|t| < \pi a/2$ and $q = e^{-\pi a}$ then

$$\sum_{n=1}^{\infty}\frac{\cosh(2tn)}{n\sinh(\pi a n)} = \log(f(-q^2)) - \log\left(\theta_4(it, e^{-a\pi})\right) \tag{11}$$

Proof.
From the Jacobi Triple Product Identity (see [4]) we have

$$\theta_4(z,q) = \prod_{n=0}^{\infty}(1-q^{2n+2})(1-q^{2n-1}e^{2iz})(1-q^{2n-1}e^{-2iz}) \tag{12}$$

By taking the logarithm of both sides and expanding the logarithm of the individual terms in a power series it is simple to show (11) from (12).

Lemma 2.
Let $q = e^{-\pi\sqrt{r}}$ with r real positive

$$\phi(x) = 2\frac{d}{dx}\left(\frac{\partial}{\partial t}\log\left(\vartheta_4\left(\frac{it\pi}{2}, e^{-2\pi x}\right)\right)_{t=x}\right) \tag{13}$$

then

$$\frac{d(\sqrt{r})}{dk} = \frac{K^{(1)}(k)}{\phi\left(\frac{K(\sqrt{1-k^2})}{K(k)}\right)} = \frac{K^{(1)}(k)}{\phi\left(\frac{K(k')}{K(k)}\right)} \tag{14}$$

Where $K^{(1)}(k)$ is the first derivative of K.
Proof.
From Lemma 1 we have

$$2\frac{\partial}{\partial t}\log\left(\vartheta_4\left(\frac{it\pi}{2},e^{-2\pi x}\right)\right)_{t=x}=-\pi\sum_{n=1}^{\infty}\frac{1}{\cosh\left(n\pi x\right)}=\frac{\pi}{2}-K(k_x)$$

then

$$\sqrt{x(k_2)}-\sqrt{x(k_1)}=-\int_{k_1}^{k_2}\frac{K^{(1)}(k)}{\phi\left(\frac{K\left(\sqrt{1-k^2}\right)}{K(k)}\right)}dk$$

Differentiating the above relation with respect to k we get the result.

Lemma 3.
Set $q=e^{-\pi\sqrt{r}}$ and

$$\{r,k\}:=\frac{dr}{dk}=2\frac{K(k')K^{(1)}(k)}{K(k)\phi\left(\frac{K(k')}{K(k)}\right)}$$

Then

$$\{r,k\}=\frac{\pi\sqrt{r}}{K^2(k_r)k_r k_r'^2} \tag{15}$$

Proof.
From (9) taking the logarithmic derivative with respect to k and using Lemma 2 we get:

$$\pi\left\{r,k\right\}\left(1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)=\left(\frac{1-5k^2}{(k-k^3)}+\frac{6K^{(1)}}{K}\right)\frac{4K'}{K} \tag{16}$$

But it is known that

$$\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}=\frac{1}{24}+\frac{K}{6\pi^2}((5-k^2)K-6E) \tag{17}$$

Hence

$$\{r,k\}=\frac{\pi K'}{K^2}\frac{\frac{1-5k^2}{k-k^3}+\frac{6K^{(1)}}{K}}{(k^2-5)K+6E} \tag{18}$$

Also

$$a(r)=\frac{\pi}{4K^2}+\sqrt{r}-\frac{E\sqrt{r}}{K},$$

where $a(r)$ is the elliptic alpha function. Using the above relations we get the result.
Note.
1) The first derivative of K is

$$K^{(1)}=\frac{E}{k_r\cdot k_r'^2}-\frac{K}{k_r}$$

where $k = k_r$ and $k' = k'_r = \sqrt{1-k_r^2}$.
2) In the same way we can find form the relation

$$k_{4r} = \frac{1-k'_r}{1+k'_r} \tag{19}$$

the 2-degree modular equation of the derivative.
Noting first that (the proof is easy)

$$\{r, k'_r\} = \frac{k'_r}{k_r}\{r, k_r\} \tag{20}$$

we have

$$\{r, k_{4r}\} = \frac{k'_r(1+k'_r)^2}{2k_r}\{r, k_r\} \tag{21}$$

3 The Ramanujan's Cubic Continued Fraction

Let

$$V(q) := \frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\ldots \tag{22}$$

is the Ramanujan's cubic continued fraction, then holds

Lemma 4.

$$V(q) = \frac{2^{-1/3}(k_{9r})^{1/4}(k'_r)^{1/6}}{(k_r)^{1/12}(k'_{9r})^{1/2}} \tag{23}$$

where the k_{9r} are given by (see [7]):

$$\sqrt{k_r k_{9r}} + \sqrt{k'_r k'_{9r}} = 1 \tag{24}$$

Proof.
The proof can be found in [18].

Lemma 5.
If

$$G(x) = \frac{x}{\sqrt{2\sqrt{x} - 3x + 2x^{3/2} - 2\sqrt{x}\sqrt{1 - 3\sqrt{x} + 4x - 3x^{3/2} + x^2}}}$$

and

$$k_r = G(w) \tag{25}$$

then

$$k_{9r} = \frac{w}{k_r}$$

and

$$k'_{9r} = \frac{(1-\sqrt{w})^2}{k'_r}$$

Proof.
See [18].

Theorem 1.
Set $T = \sqrt{1 - 8V^3(q)}$ then

$$(k_r)^2 = \frac{(1-T)(3+T)^3}{(1+T)(3-T)^3} \tag{26}$$

Proof.
See [18].

Equation (26) is a solvable quartic equation with respect to T.
An example of evaluation is

$$V(e^{-\pi}) = \frac{1}{2}\left(-2 - \sqrt{3} + \sqrt{3(3+2\sqrt{3})}\right) \tag{27}$$

Main Theorem.
Let $q = e^{-\pi\sqrt{r}}$, then

$$V'(q) = \frac{dV(q)}{dq} = \frac{-2\sqrt{r}}{q\pi}\frac{dV}{dr} = \frac{4K^2(k_r)k_r'^2(V(q)+V^4(q))}{3q\pi^2\sqrt{r}\sqrt{1-8V^3(q)}} \tag{28}$$

Proof.
Derivate (26) with respect to r then

$$\sqrt{\frac{2k_r}{\{k,r\}}} = \frac{4T(3+T)}{(3-T)^2(1+T)}\sqrt{\frac{dT}{dr}} \tag{29}$$

or

$$T_r = \frac{dT}{dr} = \frac{1}{8k_r\{r,k\}}\frac{(9-T^2)(1-T^2)}{T^2} \tag{30}$$

Using the relation $T = \sqrt{1 - 8V(q)^3}$, we get

$$\frac{dV(q)}{dr} = -\frac{2}{3}\frac{V(q)+V^4(q)}{k_r\{r,k\}\sqrt{1-8V^3(q)}} \tag{a}$$

which is the result.
Hence the problem of finding $V(q)$ and $V'(q)$ is completely solvable in radicals when we know k_r and $K(k_r)$ (see [12]), $r \in \mathbf{Q}$, $r > 0$.

We often use the notations $V[r] := V(e^{-\pi\sqrt{r}})$, $T[r] := T(e^{-\pi\sqrt{r}}) = t$.

Proposition 1.

$$V[4r] = \frac{1-T[r]}{4V[r]} \tag{31}$$

Proof.
See [9].

Proposition 2.
Set $T'[4r] = u$, $T'[r] = \nu$, then

$$\frac{u}{\nu} = \frac{(1-t)(3+t)}{8\sqrt{t}(1+t)^{5/3}(3-t)^{1/2}} \tag{32}$$

Proof.
From (19), (20), (21) and (28) we get

$$V'[4r] = \frac{-2\{k,r\}}{3\frac{1-k'}{1+k'}\frac{k'(1+k')^2}{2k}}\frac{V[4r]+V[4r]^4}{T[r]} \tag{33}$$

If we use the duplication formula (31) we get the result.

Evaluations.
1) We can calculate now easy the values of $V'(q)$ from (28) using (26). An example of evaluation is

$$k_1 = \frac{1}{\sqrt{2}}$$

,

$$E(k_1) = \frac{4\pi^{3/2}}{\Gamma(-1/4)^2} + \frac{\Gamma(3/4)^2}{2\sqrt{\pi}}$$

and

$$K(k_1) = \frac{8\pi^{3/2}}{\Gamma(-1/4)^2}$$

When $r = 1$ we get

$$\{r,k\} = \frac{8\sqrt{2}\Gamma(3/4)^4}{\pi^2}$$

Hence

$$V'(e^{-\pi}) = -\frac{64\left(-26-15\sqrt{3}+10\sqrt{3+2\sqrt{3}}+6\sqrt{9+6\sqrt{3}}\right)}{\sqrt{45+26\sqrt{3}-18\sqrt{3+2\sqrt{3}}-10\sqrt{9+6\sqrt{3}}}}\frac{e^{\pi}\pi}{\Gamma\left(-\frac{1}{4}\right)^4} \tag{34}$$

2) It is

$$T_1 = T(e^{-\pi\sqrt{3}}) = -39+22\sqrt{3} - \frac{2\cdot 6^{2/3}(-123+71\sqrt{3})}{\left(-4725+2728\sqrt{3}-\sqrt{4053-2340\sqrt{3}}\right)^{1/3}} +$$

$$+2 \cdot 6^{1/3} \left(-4725 + 2728\sqrt{3} - \sqrt{4053 - 2340\sqrt{3}} \right)^{1/3}$$

and $V_1 = V(e^{-\pi\sqrt{3}}) = \frac{1}{2}\sqrt[3]{1-T_1^2}$.
From tables and (15) it is:

$$\{3, k_3\} = \frac{192\sqrt{2}\left(-1+\sqrt{3}\right)\pi^2}{\Gamma\left(\frac{1}{6}\right)^2 \Gamma\left(\frac{1}{3}\right)^2}$$

We find the value of $V'(e^{-\pi\sqrt{3}})$ in terms of Gamma function and algebraic numbers.

$$V'(e^{-\pi\sqrt{3}}) = \frac{4\sqrt{3}e^{\pi\sqrt{3}}}{3} \frac{V_1 + V_1^4}{k_3\{3, k_3\}\sqrt{1-8V_1^3}}$$

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